New Perturbative Solutions of the Kerr–Newman Dilatonic Black Hole Field Equations

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Abstract

This work describes new perturbative solutions to the classical, four-dimensional Kerr–Newman dilaton black hole field equations. Our solutions do not require the black hole to be slowly rotating. The unperturbed solution is taken to be the ordinary Kerr solution, and the perturbation parameter is effectively the square of the charge-to-mass ratio $(Q/M)^2$ of the Kerr–Newman black hole. We have uncovered a new, exact conjugation (mirror) symmetry for the theory, which maps the small coupling sector to the strong coupling sector $(\phi \to -\phi)$. We also calculate the gyromagnetic ratio of the black hole. 4.60.+n, 11.17.+y, 97.60.lf

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I. INTRODUCTION

In recent years a great deal of effort has gone into the study of black hole solutions in both the classical and the quantum theory of gravity. In particular the role of black holes in the quantized theory of gravity has been extensively investigated, but no consensus has been reached on what this role is.

In several recent articles [1–8], we have analyzed this problem, especially in the context of the Schwarzschild black hole solution, and have come to the conclusion that quantum black holes can be identified as particle excitations of quantum extended objects such as p-branes or strings [1]. From this point of view quantum black holes should be treated as particles possessing mass, charge and spin. In addition to these properties, theories of extended objects have gravitational sectors with both tensor and scalar fields. So a consistent treatment of the gravitational interactions of such objects includes the latter, the dilaton field, as well as the former.

Our objective is to find the general classical solutions to the field equations in four dimensions for a massive, charged, and spinning black hole interacting with a dilaton. The proper action for such a system is the Einstein–Maxwell action enriched with a dilaton field. The solutions of the field equations are parametrized by the mass M, the charge Q, the (spin) angular momentum J and the dilaton parameter a.

At present there is no known solution for an arbitrary value of a. Exact solutions have been found, however, for the cases $a = 0$ and $a = \sqrt{3}$. While the former is just the classic Kerr–Newman solution [9], the latter is called the Kaluza–Klein solution since it can be shown to be equivalent to a compactification of the 5-dimensional vacuum Einstein equations [10,11]. Another exact solution has been found by Sen [12], but this solution requires the incorporation of an additional axion field into the action. Perturbative solutions are known in the case of arbitrary a [10,13]. The perturbation parameter is the black hole angular momentum J per unit mass, and the corresponding solutions are for slowly rotating black holes.

Our perturbative expansion, on the other hand, is performed in terms of the charge-to-mass ratio of the black hole and therefore is not restricted to the slowly rotating approximation. The unperturbed solution in our method is the neutral four-dimensional Kerr space–time [9].

In Section II below we present a derivation of the relevant forms for the field equations of the dilaton Kerr–Newman problem. Our notation follows that of Chandrasekhar [9] very closely, except that our convention is $eta_{ij} = \text{diag}(-1, 1, 1, 1)$. In Section III we discuss an exact conjugation (mirror) symmetry of the field equations, which maps small to strong gravitational coupling, $\phi \to -\phi$ [14]. In Section IV we do perturbation theory about small electric charge Q^2 and obtain explicit solutions to order $O(Q^4)$. In Section V we compare our results to those obtained in the slowly rotating approximation, including a calculation of the black hole gyromagnetic ratio [10,15].

II. DERIVATION OF THE FIELD EQUATIONS

The action describing the Einstein–Maxwell theory interacting with a dilaton in four dimensions is $(G = 1)$,

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[R - \frac{1}{2}(\nabla\phi)^2 - e^{-a\phi}F^2 \right] , \qquad (2.1)$$

where R is the scalar curvature, ϕ is the dilaton field and F_{ij} is the Maxwell field. In tensor components, the corresponding field equations are

$$\begin{cases} \nabla^i(e^{-a\phi}F_{ij}) = 0 \\ \qquad\qquad\qquad\qquad \text{(Maxwell)} \\ \nabla_{[k}F_{ij]} = 0 , \end{cases} \qquad (2.2)$$

$$\nabla^2\phi = -ae^{-a\phi}F^2 \quad \text{(dilaton)} , \qquad (2.3)$$

and

$$R_{ij} = \frac{1}{2}\nabla_i\phi\nabla_j\phi + 2T_{ij}^{EM} \quad \text{(Einstein)} , \qquad (2.4)$$

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where R_{ij} is the Ricci tensor, and the electromagnetic energy–momentum tensor is given as

$$T_{ij}^{EM} = e^{-a\phi} \left[F_{ik}F_j^k - \frac{1}{4}g_{ij}F^2 \right] \ . \tag{2.5}$$

The general form for the metric which describes a stationary axisymmmetric space–time is [9]

$$ds^2 = -e^{2\nu}(dt)^2 + e^{2\psi}(d\varphi - \omega dt)^2 + e^{2\mu_2}(dr)^2 + e^{2\mu_3}(d\theta)^2 \ , \tag{2.6}$$

in which $\nu, \psi, \omega, \mu_2,$ and μ_3 are functions of r and θ only. Following Chandrasekhar [9], we shall work in the special frame of an inertial local coordinate system (tetrad frame) with the basis 1-forms

$$\Omega^0 = e^\nu dt \ ; \quad \Omega^1 = e^\psi(d\varphi - \omega dt)$$

$$\Omega^2 = e^{\mu_2} dr \ ; \quad \Omega^3 = e^{\mu_3} d\theta \ . \tag{2.7}$$

The coordinates are

$$x^0 = t \ ; \quad x^1 = \varphi \ ; \quad x^2 = r \ ; \quad x^3 = \theta \ . \tag{2.8}$$

The main advantage of a tetrad frame is that it is locally flat:

$$e_a^j e_{jb} = \eta_{ab} = \mathrm{diag}(-1, 1, 1, 1) \tag{2.9}$$

The vierbein field corresponding to the frame of Eq.(2.7) is

$$e_0^i = (e^{-\nu}, \omega e^{-\nu}, 0, 0)$$

$$e_1^i = (0, e^{-\psi}, 0, 0)$$

$$e_2^i = (0, 0, e^{-\mu_2}, 0)$$

$$e_3^i = (0, 0, 0, e^{-\mu_3}) \tag{2.10}$$

and

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$$e_{0i} = (-e^\nu, 0, 0, 0)$$

$$e_{1i} = (-\omega e^\psi, e^\psi, 0, 0)$$

$$e_{2i} = (0, 0, e^{\mu_2}, 0)$$

$$e_{3i} = (0, 0, 0, e^{\mu_3}) \ . \tag{2.11}$$

Note that latin indices i, j, k are tensor indices, while a, b, c are tetrad indices.

In the calculations which follow we shall find it convenient to define

$$f_{ab} \equiv F_{ab} e^{-a\phi} \ , \tag{2.12}$$

where F_{ab} and ϕ are by axisymmetry functions of r and θ only. Maxwell's equations (2.2) in the tetrad basis take the following form when we group the equations which contain only total derivatives

$$(e^{\psi+\mu_2} F_{12})_{,3} - (e^{\psi+\mu_3} F_{13})_{,2} = 0$$

$$(e^{\psi+\mu_3} f_{02})_{,2} + (e^{\psi+\mu_2} f_{03})_{,3} = 0 \tag{2.13a}$$

and

$$(e^{\nu+\mu_3} f_{12})_{,2} + (e^{\nu+\mu_2} f_{13})_{,3} = e^{\psi+\mu_3} f_{02}\omega_{,2} + e^{\psi+\mu_2} f_{03}\omega_{,3}$$

$$(e^{\nu+\mu_2} F_{02})_{,3} - (e^{\nu+\mu_3} F_{03})_{,2} = e^{\psi+\mu_2} F_{12}\omega_{,3} - e^{\psi+\mu_3} F_{13}\omega_{,2} \ . \tag{2.13b}$$

All other components vanish.

Equations (2.13a) are just integrability conditions which allow us to define "potentials" A and B as

$$e^{\psi+\mu_2} F_{12} = A_{,2} \ ; \ e^{\psi+\mu_3} F_{13} = A_{,3}$$

$$e^{\psi+\mu_2} f_{03} = -B_{,2} \ ; \ e^{\psi+\mu_3} f_{02} = B_{,3} \ . \tag{2.14}$$

Inserting the expressions in Eq.(2.14) into Eqs.(2.13b) yields the following form for Maxwell's equations,

$$(e^{-\psi+\nu-\mu_2+\mu_3-a\phi}A_{,2}),_2 + (e^{-\psi+\nu+\mu_2-\mu_3-a\phi}A_{,3}),_3 = \omega,_2\, B,_3 - \omega,_3\, B,_2$$

$$(e^{-\psi+\nu-\mu_2+\mu_3+a\phi}B_{,2}),_2 + (e^{-\psi+\nu+\mu_2-\mu_3+a\phi}B_{,3}),_3 = \omega,_3\, A,_2 - \omega,_2\, A,_3 \ . \tag{2.15}$$

Again in the tetrad frame, the expression for the dilaton field (Eq.(2.3)) becomes

$$e^{-\mu_2}(e^{-\mu_2}\phi,_2),_2 + e^{-\mu_3}(e^{-\mu_3}\phi,_3),_3 + e^{-2\mu_2}\phi,_2\,(\beta,_2 +\mu_{3,2}) + e^{-2\mu_3}\phi,_3\,(\beta,_3 +\mu_{2,3})$$

$$= -2ae^{a\phi-2\psi}[e^{-2\mu_2}(A,_2{}^2 - B,_2{}^2) + e^{-2\mu_3}(A,_3{}^2 - B,_3{}^2)] \ , \tag{2.16}$$

in which β is defined as

$$\beta \equiv \psi + \nu \ . \tag{2.17}$$

The Einstein equations (2.4) and (2.5) are

$$R_{ab} = \frac{1}{2}e^{-\mu_a-\mu_b}\phi,_a\,\phi,_b + 2T_{ab} \tag{2.18}$$

where $\phi,_0 = \phi,_1 = 0$ and

$$T_{ab} = \left[\eta^{cd}f_{ac}f_{bd} - \frac{1}{4}\eta_{ab}f^2\right]e^{a\phi} \tag{2.19}$$

with

$$f^2 = f_{ab}f^{ab} = 2[-f_{02}^2 - f_{03}^2 + f_{12}^2 + f_{13}^2] \ . \tag{2.20}$$

In the tetrad basis the non-vanishing Ricci components are [9]

$$R_{00} = e^{-2\mu_2}[\nu,_2\,,_2 +\nu,_2\,(\psi+\nu-\mu_2+\mu_3),_2] + e^{-2\mu_3}[\nu,_3\,,_3 +\nu,_3\,(\psi+\nu+\mu_2-\mu_3),_3]$$

$$-\frac{1}{2}e^{2(\psi-\nu)}[e^{-2\mu_2}\omega,_2{}^2 + e^{-2\mu_3}\omega,_3{}^2] \ , \tag{2.21}$$

$$R_{11} = -e^{-2\mu_2}[\psi,_2\,,_2 +\psi,_2\,(\psi+\nu+\mu_3-\mu_2),_2] - e^{-2\mu_3}[\psi,_3\,,_3 +\psi,_3\,(\psi+\nu+\mu_2-\mu_3),_3]$$

$$-\frac{1}{2}e^{2(\psi-\nu)}[e^{-2\mu_3}\omega,_3{}^2 + e^{-2\mu_2}\omega,_2{}^2] \ , \tag{2.22}$$

$$R_{22} = -e^{-2\mu_2}[(\psi+\nu+\mu_3),_2\,,_2 +\psi,_2\,(\psi-\mu_2),_2 +\mu_{3,2}\,(\mu_3-\mu_2),_2 +\nu,_2\,(\nu-\mu_2),_2]$$

$$-e^{-2\mu_3}[\mu_{2,3}\,,_3 +\mu_{2,3}\,(\psi+\nu+\mu_2-\mu_3),_3] + \frac{1}{2}e^{2(\psi-\nu-\mu_2)}\omega,_2{}^2 \ , \tag{2.23}$$

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$$R_{33} = -e^{-2\mu_3}[(\psi + \nu + \mu_2),_{3},_{3} + \psi,_{3}(\psi - \mu_3),_{3} + \mu_2,_{3}(\mu_2 - \mu_3),_{3} + \nu,_{3}(\nu - \mu_3),_{3}]$$
$$-e^{-2\mu_2}[\mu_{3,2},_{2} + \mu_{3,2}(\psi + \nu + \mu_3 - \mu_2),_{2}] + \frac{1}{2}e^{2(\psi-\nu-\mu_3)}\omega,_{3}^{2}, \qquad (2.24)$$

$$R_{01} = \frac{1}{2}e^{-2\psi-\mu_2-\mu_3}[(e^{3\psi-\nu-\mu_2+\mu_3}\omega,_{2}),_{2} + (e^{3\psi-\nu-\mu_3+\mu_2}\omega,_{3}),_{3}], \qquad (2.25)$$

and

$$R_{23} = -e^{-\mu_2-\mu_3}[(\psi + \nu),_{2},_{3} - (\psi + \nu),_{2}\mu_{2,3} - (\psi + \nu),_{3}\mu_{3,2} + \psi,_{2}\psi,_{3} + \nu,_{2}\nu,_{3}]$$
$$+\frac{1}{2}e^{2\psi-2\nu-\mu_2-\mu_3}\omega,_{2}\omega,_{3}. \qquad (2.26)$$

Multiplying the R_{00} and R_{11} components by $e^{\beta+\mu_2+\mu_3}$ and making use of Eq.(2.18), we arrive at the relations

$$[e^{\mu_3-\mu_2}(e^{\beta}),_{2}],_{2} + [e^{\mu_2-\mu_3}(e^{\beta}),_{3}],_{3} = 0, \qquad (2.27)$$

and

$$[e^{\beta+\mu_3-\mu_2}(\psi - \nu),_{2}],_{2} + [e^{\beta+\mu_2-\mu_3}(\psi - \nu),_{3}],_{3} = -e^{3\psi-\nu}[e^{\mu_3-\mu_2}\omega,_{2}^{2} + e^{\mu_2-\mu_3}\omega,_{3}^{2}]$$
$$-2e^{-\psi+\nu}[e^{\mu_3-\mu_2}(e^{-a\phi}A,_{2}^{2} + e^{a\phi}B,_{2}^{2})$$
$$+e^{\mu_2-\mu_3}(e^{-a\phi}A,_{3}^{2} + e^{a\phi}B,_{3}^{2})]. \qquad (2.28)$$

Eq.(2.27) is essentially the sum $R_{00} + R_{11}$ while Eq.(2.28) is the difference $R_{00} - R_{11}$.

Corresponding relations can be found for the sum and difference of R_{22} and R_{33}. Making use of the Einstein relation (Eq.(2.18)) and Eqs.(2.23) - (2.24), the difference is

$$4e^{\mu_3-\mu_2}(\beta,_{2}\mu_{3,2} + \psi,_{2}\nu,_{2}) - 4e^{\mu_2-\mu_3}(\beta,_{3}\mu_{2,3} + \psi,_{3}\nu,_{3})$$
$$= 2e^{-\beta}\left\{[e^{\mu_3-\mu_2}(e^{\beta}),_{2}],_{2} - [e^{\mu_2-\mu_3}(e^{\beta}),_{3}],_{3}\right\}$$
$$-e^{2(\psi-\nu)}[e^{\mu_3-\mu_2}\omega,_{2}^{2} - e^{\mu_2-\mu_3}\omega,_{3}^{2}]$$
$$+(e^{-2\mu_2}\phi,_{2}^{2} - e^{-2\mu_3}\phi,_{3}^{2})e^{\mu_2+\mu_3}$$
$$+4e^{-2\psi}[e^{\mu_3-\mu_2}(e^{-a\phi}A,_{2}^{2} + e^{a\phi}B,_{2}^{2})$$
$$-e^{\mu_2-\mu_3}(e^{-a\phi}A,_{3}^{2} + e^{a\phi}B,_{3}^{2})], \qquad (2.29)$$

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while the sum is

$$R_{22} + R_{33} = \frac{1}{2}(e^{-2\mu_2}\phi_{,2}^{2} + e^{-2\mu_3}\phi_{,3}^{2}) \ . \tag{2.30}$$

Finally, the R_{01} component leads to the relation,

$$(e^{3\psi-\nu-\mu_2+\mu_3}\omega_{,2})_{,2} + (e^{3\psi-\nu+\mu_2-\mu_3}\omega_{,3})_{,3} = 4(A_{,2}\,B_{,3} - A_{,3}\,B_{,2}) \ , \tag{2.31}$$

while the R_{23} component gives

$$\beta_{,2\,,3} - \beta_{,2}\,\mu_{2,3} - \beta_{,3}\,\mu_{3,2} + \psi_{,2}\,\psi_{,3} + \nu_{,2}\,\nu_{,3}$$
$$= \frac{1}{2}e^{2(\psi-\nu)}\omega_{,2}\,\omega_{,3} - \frac{1}{2}\phi_{,2}\,\phi_{,3} - 2e^{-2\psi}(e^{-a\phi}A_{,2}\,A_{,3} + e^{a\phi}B_{,2}\,B_{,3}) \ . \tag{2.32}$$

The final form for the field equations is arrived at by means of the definitions,

$$\chi \equiv e^{-\psi+\nu} \ , \quad \Delta \equiv e^{2(\mu_3-\mu_2)}$$
$$\rho^2 \equiv e^{2\mu_3} \implies e^{2\mu_2} = \frac{\rho^2}{\Delta}$$
$$\Psi \equiv \frac{e^{\beta}}{\chi} = e^{2\psi} \implies e^{2\nu} = \chi e^{\beta} \ ; \quad (\beta = \psi + \nu) \ . \tag{2.33}$$

For convenience we also define

$$\mu \equiv \cos\theta$$
$$\delta \equiv 1 - \mu^2 = \sin^2\theta \ . \tag{2.34}$$

Eq.(2.27) is now readily solved,

$$e^{\beta(r,\theta)} = \sqrt{\delta(\theta)\Delta(r)} = \sqrt{\Delta(r)}\sin\theta \tag{2.35}$$

where $\Delta(r)$ is quadratic,

$$\Delta(r) = r^2 - 2Mr + M_0^2$$
$$= (r - r_+)(r - r_-) \tag{2.36}$$

with

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$$r_\pm = M \pm \sqrt{M^2 - M_0{}^2} \ . \tag{2.37}$$

Such a solution is also found in the case of the Kerr and Kerr–Newman problems [9]. Only the value of M_0 differs and its present value will be determined in the following section.

Recalling the definitions in Eqs.(2.33)–(2.34), we obtain the relations,

$$e^{2\psi} = \Psi = \frac{\sqrt{\delta\Delta}}{\chi} \ , \qquad e^{2\nu} = \chi\sqrt{\delta\Delta} \ . \tag{2.38}$$

The expressions

$$\partial_3 = \partial_\theta = -\sqrt{\delta}\partial_\mu \ ,$$

$$\partial_2 = \partial_r \ , \tag{2.39}$$

allow us to write the field equations as

$$\left[\frac{\Delta}{\Psi}e^{-a\phi}A_{,r}\right]_{,r} + \left[\frac{\delta}{\Psi}e^{-a\phi}A_{,\mu}\right]_{,\mu} = -\omega_{,r}\,B_{,\mu} + \omega_{,\mu}\,B_{,r}$$

$$\tag{2.40}$$

$$\left[\frac{\Delta}{\Psi}e^{a\phi}B_{,r}\right]_{,r} + \left[\frac{\delta}{\Psi}e^{a\phi}B_{,\mu}\right]_{,\mu} = \omega_{,r}\,A_{,\mu} - \omega_{,\mu}\,A_{,r} \ ,$$

for *Maxwell's equations*,

$$(\Delta\phi_{,r})_{,r} + (\delta\phi_{,\mu})_{,\mu} = -\frac{2a}{\Psi}\left[\Delta(e^{-a\phi}A_{,r}{}^2 - e^{a\phi}B_{,r}{}^2) + \delta(e^{-a\phi}A_{,\mu}{}^2 - e^{a\phi}B_{,\mu}{}^2)\right] \ , \tag{2.41}$$

for the *dilaton*, and

$$\left[\left(\frac{\Delta\Psi_{,r}}{\Psi}\right)_{,r} + \left(\frac{\delta\Psi_{,\mu}}{\Psi}\right)_{,\mu}\right] = -\frac{2}{\Psi}\left[\Delta(e^{-a\phi}A_{,r}{}^2 + e^{a\phi}B_{,r}{}^2) + \delta(e^{-a\phi}A_{,\mu}{}^2 + e^{a\phi}B_{,\mu}{}^2)\right]$$

$$- \left[\Delta\left(\frac{\Psi}{\Delta}\omega_{,\mu}\right)^2 + \delta\left(\frac{\psi}{\delta}\omega_{,r}\right)^2\right] \ , \tag{2.42}$$

for the $R_{00} - R_{11}$ component of *Einstein's equation* (2.28),

$$\left[\left(\frac{\Psi\omega_{,r}}{\delta}\right)_{,r} + \left(\frac{\Psi\omega_{,\mu}}{\Delta}\right)_{,\mu}\right] = \frac{4}{\Psi}\left[A_{,\mu}\,B_{,r} - A_{,r}\,B_{,\mu}\right]$$

$$- \left[\left(\frac{\Psi_{,r}}{\Psi}\right)\left(\frac{\Psi\omega_{,r}}{\delta}\right) + \left(\frac{\Psi_{,\mu}}{\Psi}\right)\left(\frac{\Psi\omega_{,\mu}}{\Delta}\right)\right] \ , \tag{2.43}$$

for the R_{01} component (cf. Eq.(2.31)), and finally,

$$-\frac{\Delta}{2}\left[\left(\frac{\Psi_{,r}}{\Psi}\right)^2+\left(\frac{\Psi\omega_{,\mu}}{\Delta}\right)^2\right]+\frac{\delta}{2}\left[\left(\frac{\Psi_{,\mu}}{\Psi}\right)^2+\left(\frac{\Psi\omega_{,r}}{\delta}\right)^2\right]$$

$$+\Delta_{,r}\left(\frac{\rho_{,r}}{\rho}+\frac{1}{2}\frac{\Psi_{,r}}{\Psi}\right)-\delta_{,\mu}\left(\frac{\rho_{,\mu}}{\rho}+\frac{1}{2}\frac{\Psi_{,\mu}}{\Psi}\right)$$

$$=2+\frac{1}{2}\left(\Delta\phi_{,r}^2-\delta\phi_{,\mu}^2\right)+\frac{2}{\Psi}\left[\Delta(e^{-a\phi}A_{,r}^2+e^{a\phi}B_{,r}^2)-\delta(e^{-a\phi}A_{,\mu}^2+e^{a\phi}B_{,\mu}^2)\right]\ , \qquad (2.44)$$

$$2\left[\left(\frac{\Delta\rho_{,r}}{\rho}\right)_{,r}+\left(\frac{\delta\rho_{,\mu}}{\rho}\right)_{,\mu}\right]=\Delta_{,r}\left(\frac{\rho_{,r}}{\rho}+\frac{1}{2}\frac{\Psi_{,r}}{\Psi}\right)+\delta_{,\mu}\left(\frac{\rho_{,\mu}}{\rho}+\frac{1}{2}\frac{\Psi_{,\mu}}{\Psi}\right)$$

$$-\frac{\Delta}{2}\left[\left(\frac{\Psi_{,r}}{\Psi}\right)^2-\left(\frac{\Psi\omega_{,\mu}}{\Delta}\right)^2\right]-\frac{\delta}{2}\left[\left(\frac{\Psi_{,\mu}}{\Psi}\right)^2-\left(\frac{\Psi\omega_{,r}}{\delta}\right)^2\right]$$

$$-\frac{1}{2}\left(\Delta\phi_{,r}{}^2+\delta\phi_{,\mu}{}^2\right)\ , \qquad (2.45)$$

$$\left(\frac{\Psi_{,r}}{\Psi}\right)\left(\frac{\Psi_{,\mu}}{\Psi}\right)-\left(\frac{\Psi\omega_{,r}}{\delta}\right)\left(\frac{\Psi\omega_{,\mu}}{\Delta}\right)+\frac{1}{2}\frac{\Delta_{,r}}{\Delta}\frac{\delta_{,\mu}}{\delta}-\frac{\Delta_{,r}}{\Delta}\left(\frac{\rho_{,\mu}}{\rho}+\frac{1}{2}\frac{\Psi_{,\mu}}{\Psi}\right)-\frac{\delta_{,\mu}}{\delta}\left(\frac{\rho_{,r}}{\rho}+\frac{1}{2}\frac{\Psi_{,r}}{\Psi}\right)$$

$$=-\frac{4}{\Psi}\left[e^{-a\phi}A_{,r}\,A_{,\mu}+e^{a\phi}B_{,r}\,B_{,\mu}\right]-\phi_{,r}\,\phi_{,\mu}\ , \qquad (2.46)$$

corresponding to the difference $R_{22}-R_{33}$ (Eq.(2.29)), the sum $R_{22}+R_{33}$ (Eq.(2.30)) and R_{23} (Eq.(2.32)) respectively.

In the limit $a\to 0$, the theory reproduces the field equations for the Kerr-Newman black hole [9]. The solutions of the above equations will determine the functions A,B,ϕ,Ψ,ω and ρ as functions of r and θ (x_2 and x_3).

III. MIRROR CONJUGATE SOLUTIONS

Equations (2.40)-(2.46) possess alternative solutions which are the mirror conjugates to those specified above. This is a generalization of a symmetry of Ernst's equations for a Kerr-Newman black hole [9]. To see this we define the following transformation:

$$\tilde{t}\equiv\varphi\ ;\quad \tilde{\varphi}\equiv-t$$

$$\tilde{\chi}\equiv-\frac{\chi}{\chi^2-\omega^2}\ ;\quad \tilde{\omega}\equiv\frac{\omega}{\chi^2-\omega^2}$$

$$-\tilde{A}_{,\mu}\equiv\chi\sqrt{\frac{\Delta}{\delta}}e^{-a\phi}A_{,r}+\omega B_{,\mu}$$

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$$\tilde{A}_{,r} \equiv \chi\sqrt{\frac{\delta}{\Delta}}e^{-a\phi}A_{,\mu} -\omega B_{,r}$$

$$-\tilde{B}_{,\mu} \equiv \chi\sqrt{\frac{\Delta}{\delta}}e^{a\phi}B_{,r} -\omega A_{,\mu}$$

$$\tilde{B}_{,r} \equiv \chi\sqrt{\frac{\delta}{\Delta}}e^{a\phi}B_{,\mu} +\omega A_{,r}$$

$$\tilde{\phi} \equiv -\phi \ . \tag{3.1}$$

Also we define

$$\tilde{\Psi} \equiv \frac{\sqrt{\Delta\delta}}{\tilde{\chi}} \ . \tag{3.2}$$

The quantities ρ, r, θ, Δ and δ are unchanged, and the field equations as well as the metric are invariant under this transformation. The metric is then

$$ds^2 = -e^{2\tilde{\nu}}d\tilde{t}^2 + e^{2\tilde{\psi}}(d\tilde{\varphi} - \tilde{\omega}d\tilde{t})^2 + e^{2\mu_2}dr^2 + e^{2\mu_3}d\theta^2$$

$$= -e^{2\tilde{\nu}}d\varphi^2 + e^{2\tilde{\psi}}(dt + \tilde{\omega}d\varphi)^2 + e^{2\mu_2}dr^2 + e^{2\mu_3}d\theta^2 \ . \tag{3.3a}$$

and it can be re-written as follows,

$$ds^2 = -\sqrt{\Delta\delta}\left[\chi dt^2 - \frac{1}{\chi}(d\varphi - \omega dt)^2\right] + \rho^2\left[\frac{dr^2}{\Delta} + d\theta^2\right]$$

$$= -\sqrt{\Delta\delta}\left[\tilde{\chi}d\tilde{t}^2 - \frac{1}{\tilde{\chi}}(d\tilde{\varphi} - \tilde{\omega}d\tilde{t})^2\right] + \rho^2\left[\frac{dr^2}{\Delta} + d\theta^2\right] \ , \tag{3.3b}$$

which displays its invariance property.

The invariance of the field equations under these transformations means that for each solution $\Psi, \phi, \rho, \omega, A, B$ there exists a dual solution $\tilde{\Psi}, \tilde{\phi}, \tilde{\rho} = \rho, \tilde{\omega}, \tilde{A}, \tilde{B}$. Since $\tilde{\phi} = -\phi$, the transformation in Eq.(3.1) maps the weak gravitational coupling regime to the strong regime. This is an example of mirror conjugation [14]. The tilded quantities, following the convention of Chandrasekhar [9], are the ones calculated in the following section. Eq.(3.1) can then be used to transform to the untilded quantities if desired. The inverse transformation, finally, is given as follows,

$$\varphi \equiv \tilde{t} \ ; \quad t \equiv -\tilde{\varphi}$$

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$$\chi \equiv -\frac{\tilde{\chi}}{\tilde{\chi}^2 - \tilde{\omega}^2} \; ; \quad \omega \equiv \frac{\tilde{\omega}}{\tilde{\chi}^2 - \tilde{\omega}^2}$$

$$- A_{,\mu} \equiv \tilde{\chi}\sqrt{\frac{\Delta}{\delta}}e^{-a\tilde{\phi}}\tilde{A}_{,r} + \tilde{\omega}\tilde{B}_{,\mu}$$

$$A_{,r} \equiv \tilde{\chi}\sqrt{\frac{\delta}{\Delta}}e^{-a\tilde{\phi}}\tilde{A}_{,\mu} - \tilde{\omega}\tilde{B}_{,r}$$

$$- B_{,\mu} \equiv \tilde{\chi}\sqrt{\frac{\Delta}{\delta}}e^{a\tilde{\phi}}\tilde{B}_{,r} - \tilde{\omega}\tilde{A}_{,\mu}$$

$$B_{,r} \equiv \tilde{\chi}\sqrt{\frac{\delta}{\Delta}}e^{a\tilde{\phi}}\tilde{B}_{,\mu} + \tilde{\omega}\tilde{A}_{,r}$$

$$\phi \equiv -\tilde{\phi} \ . \tag{3.4}$$

IV. PERTURBATIVE EXPANSION OF THE FIELDS

We now expand each of the fields in a power series of Q^2, starting from the neutral Kerr solution. This expansion will result in corrections to the fields in terms of the square of the charge-to-mass ratio $(Q/M)^2$. The fields we are expanding are the components of the tilded fields of the convention used by Chandrasekhar [9] as defined in Eq.(3.1a). However for convenience we drop the tilde symbol in this section. Also for convenience we introduce the following definitions

$$\rho_0^2 \equiv r^2 + \alpha^2\mu^2$$

$$\Delta_0 \equiv r^2 - 2Mr + \alpha^2$$

$$\Psi_0 \equiv -\frac{(\Delta_0 - \alpha^2\delta)}{\rho_0^2} \ , \tag{4.1}$$

where the integration constants M and α are identified from the leading order at large r with the mass and the angular momentum per unit mass of the Kerr black hole solution, and

$$\Psi \equiv \Psi_0 e^{-f} \; ; \quad \rho^2 \equiv \rho_0^2 e^g \ . \tag{4.2}$$

For later reference we note here that

$$\frac{\Psi_{,\sigma}}{\Psi} = \frac{\Psi_{0,\sigma}}{\Psi_0} - f_{,\sigma}$$

$$\frac{\rho_{,\sigma}}{\rho} = \frac{\rho_{0,\sigma}}{\rho_0} + \frac{1}{2}g_{,\sigma} \ , \tag{4.3}$$

where $\sigma = r$ or μ.

A. Series Expansion of the Field Equations

The small charge perturbation expansions of the fields are defined to be

$$\omega_{,\sigma} \equiv e^f \sum_{n=0}^{\infty} Q^{2n} \omega_\sigma^{(n)}$$

$$f_{,\sigma} \equiv \sum_{n=0}^{\infty} Q^{2n} f_{,\sigma}^{(n)}$$

$$g_{,\sigma} \equiv \sum_{n=0}^{\infty} Q^{2n} g_{,\sigma}^{(n)}$$

$$\phi_{,\sigma} \equiv \sum_{n=0}^{\infty} Q^{2n} \phi_{,\sigma}^{(n)} \ ; \quad (\sigma = r \text{ or } \mu) \ , \tag{4.4}$$

and

$$A_{,\sigma} \equiv e^{(a\phi-f)/2} \sum_{n=0}^{\infty} Q^{2n-1} A_\sigma^{(n)}$$

$$B_{,\sigma} \equiv e^{-(a\phi+f)/2} \sum_{n=0}^{\infty} Q^{2n-1} B_\sigma^{(n)}$$

$$\Delta \equiv \sum_{n=0}^{\infty} Q^{2n} \Delta^{(n)} \ ; \quad (\sigma = r \text{ or } \mu) \ . \tag{4.5}$$

When these expansions are inserted into Maxwell's equations, the dilaton equation, the Ψ equation, the ω equation and the ρ equations (Eqs.(2.40 - 2.46) respectively), the following recursion relations are obtained for the nth order expansion coefficients:

Maxwell's equations:

$$
\begin{aligned}
\sum_{l+m=n} [(\Delta^{(m)} A_r^{(l)})_{,r}] + (\delta A_\mu^{(n)})_{,\mu} &= \sum_{l+m=n} \Delta^{(m)} A_r^{(l)} \frac{\Psi_{0,r}}{\Psi_0} + \delta A_\mu^{(n)} \frac{\Psi_{0,\mu}}{\Psi_0} \\
&+ \Psi_0 \sum_{l+m=n} \left[B_r^{(l)} \omega_\mu^{(m)} - B_\mu^{(l)} \omega_r^{(m)} \right] \\
&+ \frac{1}{2} \sum_{l+m+k=n} \Delta^{(k)} A_r^{(l)} (a\phi_{,r}^{(m)} - f_{,r}^{(m)}) \\
&+ \frac{1}{2} \sum_{l+m=n} \delta A_\mu^{(l)} (a\phi_{,\mu}^{(m)} - f_{,\mu}^{(m)}) \ ,
\end{aligned}
\tag{4.6}
$$

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$$\sum_{l+m=n} [(\Delta^{(m)} B_r^{(l)})_{,r}] + (\delta B_\mu^{(n)})_{,\mu} = \sum_{l+m=n} [\Delta^{(m)} B_r^{(l)} \frac{\Psi_{0,r}}{\Psi_0}] + \delta B_\mu^{(n)} \frac{\Psi_{0,\mu}}{\Psi_0}$$

$$+ \Psi_0 \sum_{l+m=n} \left[-A_r^{(l)} \omega_\mu^{(m)} + A_\mu^{(l)} \omega_r^{(m)} \right]$$

$$- \frac{1}{2} \sum_{l+m+k=n} \left[\Delta^{(k)} B_r^{(l)} (a\phi_{,r}^{(m)} + f_{,r}^{(m)}) \right]$$

$$- \frac{1}{2} \sum_{l+m=n} \left[\delta B_\mu^{(l)} (a\phi_{,\mu}^{(m)} + f_{,\mu}^{(m)}) \right] , \qquad (4.7)$$

Dilaton equation:

$$\sum_{l+m=n} \left[(\Delta^{(m)} \phi_{,r}^{(l)})_{,r} \right] + (\delta \phi_{,\mu}^{(n)})_{,\mu} =$$

$$- \frac{2a}{\Psi_0} \left[\sum_{l+m+k-1=n} \Delta^{(k)} (A_r^{(l)} A_r^{(m)} - B_r^{(l)} B_r^{(m)}) + \sum_{l+m-1=n} \delta (A_\mu^{(l)} A_\mu^{(m)} - B_\mu^{(l)} B_\mu^{(m)}) \right] , \quad (4.8)$$

Ψ *equation:*

$$\sum_{l+m=n} \left[\Delta^{(l)} \delta \left(\frac{\Delta^{(m)} \Psi_{0,r}}{\Psi_0} \right)_{,r} \right] + \Delta^{(n)} \delta \left(\frac{\delta \Psi_{0,\mu}}{\Psi_0} \right)_{,\mu} - \sum_{l+m+k=n} \left[\Delta^{(k)} \delta (\Delta^{(m)} f_{,r}^{(l)})_{,r} \right]$$

$$- \sum_{l+m=n} \left[\Delta^{(l)} \delta (\delta f_{,\mu}^{(m)})_{,\mu} \right] = - \frac{2}{\Psi_0} \sum_{l+m+k+p-1=n} \Delta^{(p)} \delta [\Delta^{(k)} (A_r^{(l)} A_r^{(m)} + B_r^{(l)} B_r^{(m)})]$$

$$- \frac{2}{\Psi_0} \sum_{l+m+k-1=n} \Delta^{(k)} \delta [\delta (A_\mu^{(l)} A_\mu^{(m)} + B_\mu^{(l)} B_\mu^{(m)})]$$

$$- \Psi_0^2 \sum_{l+m=n} \left[\delta \omega_\mu^{(l)} \omega_\mu^{(m)} \right] - \Psi_0^2 \sum_{l+m+k=n} \left[\Delta^{(k)} \omega_r^{(l)} \omega_r^{(m)} \right] , \quad (4.9)$$

ω *equation:*

$$\sum_{l+m=n} \left[\Delta^{(m)} \left(\Psi_0 \omega_r^{(l)} \right)_{,r} \right] + \delta \left(\Psi_0 \omega_\mu^{(n)} \right)_{,\mu} = \frac{4}{\Psi_0} \sum_{l+m+k-1=n} \left[\Delta^{(k)} \delta (A_\mu^{(l)} B_r^{(m)} - A_r^{(l)} B_\mu^{(m)}) \right]$$

$$- \sum_{l+m=n} \Delta^{(m)} \left[\left(\frac{\Psi_{0,r}}{\Psi_0} \right) \left(\Psi_0 \omega_r^{(l)} \right) \right]$$

$$- \left(\frac{\Psi_{0,\mu}}{\Psi_0} \right) \delta \left(\Psi_0 \omega_\mu^{(n)} \right) + \Psi_0 \sum_{l+m=n} \left[\delta f_{,\mu}^{(l)} \omega_\mu^{(m)} \right]$$

$$+ \Psi_0 \sum_{l+m+k=n} \left[\Delta^{(k)} f_{,r}^{(l)} \omega_r^{(m)} \right] , \qquad (4.10)$$

ρ *equations:*

$$\Delta^{(n)} \delta \left(\frac{\Psi_{0,r}}{\Psi_0} \right) \left(\frac{\Psi_{0,\mu}}{\Psi_0} \right) - \sum_{l+m=n} \Delta^{(l)} \delta \left[\left(\frac{\Psi_{0,r}}{\Psi_0} \right) f_{,\mu}^{(m)} + \left(\frac{\Psi_{0,\mu}}{\Psi_0} \right) f_{,r}^{(m)} \right] + \frac{\Delta_r^{(n)}}{2} \delta_{,\mu}$$

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$$+ \sum_{l+m+k=n} \left[\Delta^{(k)} \delta f_{,r}^{(l)} \, f_{,\mu}^{(m)} \right] - \Psi_0^2 \sum_{l+m=n} \omega_r^{(l)} \omega_\mu^{(m)} - \delta \Delta_{,r}^{(n)} \left(\frac{\rho_{0,\mu}}{\rho_0} + \frac{1}{2} \frac{\Psi_{0,\mu}}{\Psi_0} \right)$$

$$- \delta_{,\mu} \Delta^{(n)} \left(\frac{\rho_{0,r}}{\rho_0} + \frac{1}{2} \frac{\Psi_{0,r}}{\Psi_0} \right) - \frac{1}{2} \sum_{l+m=n} \left[\delta \Delta_{,r}^{(m)} (g_{,\mu}^{(l)} - f_{,\mu}^{(l)}) + \delta_{,\mu} \Delta^{(m)} (g_{,r}^{(l)} - f_{,r}^{(l)}) \right]$$

$$= -\frac{4}{\Psi_0} \sum_{l+m+k-1=n} \Delta^{(k)} \delta (A_\mu^{(l)} A_r^{(m)} + B_r^{(l)} B_\mu^{(m)}) - \sum_{l+m+k=n} \delta \Delta^{(k)} \phi_{,r}^{(l)} \, \phi_{,\mu}^{(m)} \qquad (4.11\text{a})$$

$$\Delta^{(n)} \delta \left[-2 + \frac{\delta}{2} \left(\frac{\Psi_{0,\mu}}{\Psi_0} \right)^2 - \delta_{,\mu} \left(\frac{1}{2} \frac{\rho_{0,\mu}}{\rho_0} \frac{\Psi_{0,\mu}}{\Psi_0} \right) \right]$$

$$+ \sum_{l+m=n} \delta \Delta^{(m)} \left[-\frac{\Delta^{(l)}}{2} \left(\frac{\Psi_{0,r}}{\Psi_0} \right)^2 + \Delta_{,r}^{(l)} \left(\frac{\rho_{0,r}}{\rho} + \frac{1}{2} \frac{\Psi_{0,r}}{\Psi_0} \right) \right]$$

$$+ \sum_{l+m+k=n} \Delta^{(m)} \delta \Delta^{(k)} f_{,r}^{(l)} \left(\frac{\Psi_{0,r}}{\Psi_0} \right) - \sum_{l+m=n} \Delta^{(m)} \delta^2 f_{,\mu}^{(l)} \left(\frac{\Psi_{0,\mu}}{\Psi_0} \right)$$

$$+ \frac{1}{2} \left[\sum_{l+m+k=n} \Delta^{(k)} \delta \Delta_{,r}^{(m)} (g_{,r}^{(l)} - f_{,r}^{(l)}) - \sum_{l+m=n} \Delta^{(m)} \delta \delta_{,\mu} (g_{,\mu}^{(l)} - f_{,\mu}^{(l)}) \right]$$

$$+ \frac{1}{2} \left[- \sum_{l+m+k+p=n} \Delta^{(p)} \delta \Delta^{(k)} (f_{,r}^{(l)} \, f_{,r}^{(m)}) + \sum_{l+m+k=n} \Delta^{(k)} \delta^2 (f_{,\mu}^{(l)} \, f_{,\mu}^{(m)}) \right]$$

$$+ \frac{\Psi_0^2}{2} \left[\sum_{l+m+k=n} \Delta^{(k)} \omega_r^{(l)} \omega_r^{(m)} - \sum_{l+m=n} \delta \omega_\mu^{(l)} \omega_\mu^{(m)} \right]$$

$$= \frac{1}{2} \left[\sum_{l+m+p+k=n} \Delta^{(p)} \delta \Delta^{(k)} \phi_{,r}^{(l)} \, \phi_{,r}^{(m)} - \sum_{l+m+k=n} \Delta^{(k)} \delta^2 \phi_{,\mu}^{(l)} \, \phi_{,\mu}^{(m)} \right]$$

$$+ \frac{2}{\Psi_0} \left[\sum_{l+m+k+p-1=n} \Delta^{(p)} \delta \Delta^{(k)} (A_r^{(l)} A_r^{(m)} + B_r^{(l)} B_r^{(m)}) \right.$$

$$\left. - \sum_{l+m+k-1=n} \Delta^{(k)} \delta^2 (A_\mu^{(l)} A_\mu^{(m)} + B_\mu^{(l)} B_\mu^{(m)}) \right] \qquad (4.11\text{b})$$

$$2 \left[\Delta^{(n)} \delta \left(\frac{\delta \rho_{0,\mu}}{\rho_0} \right)_{,\mu} + \sum_{l+m=n} \Delta^{(m)} \delta \left(\frac{\Delta^{(l)} \rho_{0,r}}{\rho_0} \right)_{,r} \right]$$

$$+ \left[\sum_{l+m=n} \Delta^{(m)} \delta (\delta g_{,\mu}^{(l)})_{,\mu} + \sum_{l+m+k=n} \Delta^{(k)} \delta (\Delta^{(m)} g_{,r}^{(l)})_{,r} \right]$$

$$= \Delta^{(n)} \delta \delta_{,\mu} \left(\frac{\rho_{0,\mu}}{\rho_0} + \frac{1}{2} \frac{\Psi_{0,\mu}}{\Psi_0} \right) + \sum_{l+m=n} \Delta^{(m)} \delta \Delta_{,r}^{(l)} \left(\frac{\rho_{0,r}}{\rho_0} + \frac{1}{2} \frac{\Psi_{0,r}}{\Psi_0} \right)$$

$$\frac{1}{2} \left[\sum_{l+m+k=n} \Delta^{(k)} \delta \Delta_{,r}^{(m)} (g_{,r}^{(l)} - f_{,r}^{(l)}) + \sum_{l+m=n} \Delta^{(m)} \delta \delta_{,\mu} (g_{,\mu}^{(l)} - f_{,\mu}^{(l)}) \right]$$

$$- \frac{1}{2} \left[\sum_{l+m+k+p=n} \Delta^{(p)} \delta \Delta^{(k)} \phi_{,r}^{(l)} \, \phi_{,r}^{(m)} + \sum_{l+m+k=n} \Delta^{(k)} \delta^2 \phi_{,\mu}^{(l)} \, \phi_{,\mu}^{(m)} \right]$$

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$$-\frac{1}{2}\left[\sum_{l+m=n}\Delta^{(m)}\delta\Delta^{(l)}\left(\frac{\Psi_{0,r}}{\Psi_0}\right)^2+\Delta^{(n)}\delta^2\left(\frac{\Psi_{0,\mu}}{\Psi_0}\right)^2\right]$$

$$+\sum_{l+m+k=n}\Delta^{(k)}\delta\Delta^{(m)}\left(\frac{\Psi_{0,r}}{\Psi_0}\right)f_{,r}^{(l)}+\sum_{l+m=n}\Delta^{(m)}\delta^2\left(\frac{\Psi_{,\mu}}{\Psi}\right)f_{,\mu}^{(l)}$$

$$-\frac{1}{2}\left[\sum_{l+m+k+p=n}\Delta^{(p)}\delta\Delta^{(k)}f_{,r}^{(l)}f_{,r}^{(m)}+\sum_{n=l+m+k}\Delta^{(k)}\delta^2 f_{,\mu}^{(l)}f_{,\mu}^{(m)}\right]$$

$$+\frac{\Psi_0^2}{2}\left[\sum_{l+m=n}\delta\omega_\mu^{(l)}\omega_\mu^{(m)}+\sum_{l+m+k=n}\Delta^{(k)}\omega_r^{(l)}\omega_r^{(m)}\right] \tag{4.11c}$$

B. $n=0$ Solutions

Let us now find the solutions of these equations can be obtained for the leading orders in the charge-to-mass ratio. For $n=0$ we have the Kerr black hole characterized as follows

$$A_\sigma^{(0)}=B_\sigma^{(0)}=0 \tag{4.12}$$

$$\phi^{(0)}=0\,, \tag{4.13}$$

as well as

$$f^{(0)}=g^{(0)}=0\,, \tag{4.14}$$

and

$$\omega_r^{(0)}=-\frac{2\alpha\delta M(r^2-\alpha^2\mu^2)}{(\Delta_0-\alpha^2\delta)^2}$$

$$\omega_\mu^{(0)}=-\frac{2\alpha\Delta_0 M\mu r}{(\Delta_0-\alpha^2\delta)^2}$$

$$\Delta^{(0)}=\Delta_0\,. \tag{4.15}$$

where Δ_0 has been defined in Eq.(4.1).

C. $n=1$ Corrections

At order $n=1$ the solutions of Maxwell's equations give

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$$A_r^{(1)} = \frac{(-r^2 + \alpha^2\mu^2)}{\rho_0^4} \; ; \quad A_\mu^{(1)} = -\frac{2\alpha^2\mu r}{\rho_0^4}$$

$$\alpha B_r^{(1)} = -A_\mu^{(1)} \; ; \quad B_\mu^{(1)} = \alpha A_r^{(1)} \; . \tag{4.16}$$

From the dilaton equation we find

$$\phi_{,\sigma}^{(1)} = -\frac{a}{M} A_\sigma^{(1)} \; ; \quad (\sigma = r \text{ or } \mu) \; . \tag{4.17}$$

To obtain the remaining quantities we begin with the assumption that $\Delta^{(1)} = \Delta^{(1)}(a) = $ constant. At first we assume the following ansatz

$$f^{(1)} = \Delta^{(1)}\hat{f}^{(1)} \; ; \quad g^{(1)} = \Delta^{(1)}\hat{g}^{(1)} \; ; \quad \omega_\sigma^{(1)} = \Delta^{(1)}\hat{\omega}_\sigma^{(1)} \; . \tag{4.18}$$

Now when we set the dilaton parameter $a = 0$ in the field equations (2.2)-(2.5), we recover the Kerr–Newman space–time for which an exact solution is known [9]. For such a case, we therefore have $\Delta^{(1)}(a = 0) = 1$, $\Delta^{(n)} = 0$ $(n > 1)$ and the quantities $\hat{f}^{(1)}, \hat{g}^{(1)}, \hat{\omega}_\sigma^{(1)}, \omega_\sigma^{(0)}, A_\sigma^{(1)}$ and $B_\sigma^{(1)}$ are order $n = 1$ (i.e. Q^2) solutions of the Kerr–Newman field equations. The exact solutions for the Kerr–Newman case are

$$\hat{\Psi} = -\frac{[(\Delta_0 + Q^2) - \alpha^2\delta]}{\rho_0^2} \equiv \Psi_0 e^{-\hat{f}} \; ; \quad \hat{f} = -\ln\left(1 - \frac{Q^2}{\rho_0^2\Psi_0}\right) \; ,$$

$$\hat{\rho}^2 = r^2 + \alpha^2\mu^2 \equiv \rho_0^2 e^{\hat{g}} \; ; \quad \hat{g} = 0 \tag{4.19}$$

$$\hat{A}_{,\sigma} = QA_\sigma^{(1)} \; ; \quad \hat{B}_{,\sigma} = QB_\sigma^{(1)} \; , \tag{4.20}$$

$$\hat{\omega} = -\alpha\delta[1 + \hat{\Psi}^{-1}]$$

$$\hat{\Psi}\hat{\omega}_{,r} = \alpha\delta\left[\frac{\Psi_{0,r}}{\Psi_0} - \hat{f}_{,r}\right]$$

$$\hat{\Psi}\hat{\omega}_{,\mu} = 2\alpha\mu[1 + \hat{\Psi}] + \alpha\delta\left[\frac{\Psi_{0,\mu}}{\Psi_0} - \hat{f}_{,\mu}\right] \; . \tag{4.21}$$

The functions $\hat{f}, \hat{g}$ and $\hat{\omega}_{,\sigma}$ can be expanded as was done for their counterparts in Eq.(4.4). From Eqs.(4.9) with $a = 0$, we must have

$$\hat{f}^{(1)} = \frac{1}{\rho_0^2\Psi_0}$$

$$\hat{g}^{(1)} = 0 \tag{4.22}$$

and

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$$\Psi_0 \hat{\omega}_r^{(1)} = -\alpha \delta \hat{f}_{,r}^{(1)}$$

$$\Psi_0 \hat{\omega}_\mu^{(1)} = -2\alpha \mu \Psi_0 \hat{f}^{(1)} - \alpha \delta \hat{f}_{,\mu}^{(1)} \tag{4.23}$$

On the other hand, if we set $\Delta^{(1)} = 0$, which corresponds to the Kaluza–Klein case $a^2 = 3$ [10,11], we find the solutions

$$\bar{f}^{(1)} = \bar{g}^{(1)} = \frac{A^{(1)}}{M}$$

$$\Psi_0 \bar{\omega}_r^{(1)} = -\alpha \delta \bar{f}_{,r}^{(1)}$$

$$\Psi_0 \bar{\omega}_\mu^{(1)} = -\frac{\Delta_0 \bar{f}_{,\mu}^{(1)}}{\alpha} \ , \tag{4.24}$$

where $A^{(1)} \equiv r/\rho_0^2$. Thus for general $\Delta^{(1)} = \Delta^{(1)}(a)$, we attempt solutions of the form

$$f^{(1)} = \frac{\Delta^{(1)}}{\rho_0^2 \Psi_0} + \lambda_0 \frac{A^{(1)}}{M} \ ; \quad g^{(1)} = \lambda_0 \frac{A^{(1)}}{M}$$

$$\Psi_0 \omega_r^{(1)} = -\alpha \delta f_{,r}^{(1)}$$

$$\Psi_0 \omega_\mu^{(1)} = -2\alpha \mu \Psi_0 f^{(1)} - \alpha \delta f_{,\mu}^{(1)} \ , \tag{4.25}$$

where λ_0 is a function of a to be determined, as is $\Delta^{(1)}$. Substituting these expressions into the recursion formulas of subsection A. for $n = 1$, and modding out the Kerr-Newman part, we find

$$\Delta^{(1)}(a) = 1 - \lambda_0 \ . \tag{4.26}$$

Note that this form reproduces the two previous limiting cases $a = 0$ and $a = \sqrt{3}$ when $\lambda_0 = a^2/3$. The proof of this conjecture will be presented at the order $n = 2$ calculation in the next subsection.

In summary, the contributions to the fields at this order are given as

$$A_r^{(1)} = \frac{(-r^2 + \alpha^2 \mu^2)}{\rho_0^4} \ ; \quad A_\mu^{(1)} = -\frac{2\alpha^2 \mu r}{\rho_0^4}$$

$$\alpha B_r^{(1)} = -A_\mu^{(1)} \ ; \quad B_\mu^{(1)} = \alpha A_r^{(1)}$$

$$\phi_{,\sigma}^{(1)} = -\frac{a A_\sigma^{(1)}}{M} \ ; \quad (\sigma = r \ \text{or} \ \mu)$$

$$f^{(1)} = \frac{\Delta^{(1)}}{\rho_0^2 \Psi_0} + \lambda_0 \frac{A^{(1)}}{M} \ ; \quad g^{(1)} = \lambda_0 \frac{A^{(1)}}{M}$$

$$\Psi_0 \omega_r^{(1)} = -\alpha \delta f_{,r}^{(1)} \ ; \quad \Psi_0 \omega_\mu^{(1)} = -2\alpha \mu \Psi_0 f^{(1)} - \alpha \delta f_{,\mu}^{(1)} \ , \tag{4.27}$$

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with $\Delta^{(1)}$ given by Eq.(4.26).

D. $n = 2$ Corrections

We have also obtained the order $n = 2$ (i.e. Q^4) corrections to the electromagnetic fields. To obtain these corrections we begin by noting that the exact solutions for the Kerr–Newman case are:

$$\hat{A}_{,r} = Q\frac{(-r^2 + \alpha^2\mu^2)}{\rho_0^4} = e^{-\hat{f}/2} \sum_{n=0}^{\infty} Q^{2n-1} \hat{A}_r^{(n)}$$
$$\hat{A}_{,\mu} = -Q\frac{2\alpha^2\mu r}{\rho_0^4} = e^{-\hat{f}/2} \sum_{n=0}^{\infty} Q^{2n-1} \hat{A}_\mu^{(n)}$$
$$\hat{B}_{,\sigma} = e^{-\hat{f}/2} \sum_{n=0}^{\infty} Q^{2n-1} \hat{B}_\sigma^{(n)} ; \quad (\sigma = r \text{ or } \mu) . \tag{4.28}$$

From Eqs.(4.16) and (4.20) we have

$$\hat{A}_{,\sigma} = QA_\sigma^{(1)} ; \quad \hat{B}_{,\sigma} = QB_\sigma^{(1)} , \tag{4.29}$$

which imply that

$$\hat{A}_\sigma^{(2)} = \tfrac{1}{2}\hat{f}^{(1)} A_\sigma^{(1)}$$
$$\hat{B}_\sigma^{(2)} = \tfrac{1}{2}\hat{f}^{(1)} B_\sigma^{(1)} , \tag{4.30}$$

where again $\hat{f}^{(1)} = 1/\rho_0^2\Psi_0$ (Eq.(4.22)). This suggests that we try the following ansatz

$$A_\sigma^{(2)} = \Delta^{(1)} \hat{A}_\sigma^{(2)} + \lambda_0 c A_\sigma^{(1)}$$
$$B_\sigma^{(2)} = \Delta^{(1)} \hat{B}_\sigma^{(2)} + \lambda_0 h B_\sigma^{(1)} . \tag{4.31}$$

Substituting these expressions into Maxwell's equations (Eqs.(4.6,4.7)) for $n = 2$ and modding out the Kerr–Newman part, the equations are satisfied when c and h are given as

$$c = -\frac{\Psi_0}{4M^2}\left(1 + \frac{a^2}{\lambda_0}\right)$$
$$h = \frac{\Psi_0}{4M^2}\left(1 - \frac{a^2}{\lambda_0}\right) , \tag{4.32}$$

provided that

$$\lambda_0 = \frac{a^2}{3} ,$$
(4.33)

which is the expected form. The order $n = 2$ corrections for the A and B potentials are therefore

$$A_\sigma^{(2)} = \left(\frac{\Delta^{(1)}}{2\rho_0^2\Psi_0} - \frac{a^2\Psi_0}{3M^2} \right) A_\sigma^{(1)}$$
$$B_\sigma^{(2)} = \left(\frac{\Delta^{(1)}}{2\rho_0^2\Psi_0} - \frac{a^2\Psi_0}{6M^2} \right) B_\sigma^{(1)} ,$$
(4.34)

with $\Delta^{(1)} = 1 - a^2/3$.

E. Fields and Metric Tensor Elements

In this section we summarize the expressions for the electric, magnetic and dilaton fields and the expressions for the tensor elements. Using the definitions

$$C_A \equiv \frac{\Delta^{(1)}}{2\,\rho_0^2\,\Psi_0} - \frac{a^2\,\Psi_0}{3\,M^2}$$
$$C_B \equiv \frac{\Delta^{(1)}}{2\,\rho_0^2\,\Psi_0} - \frac{a^2\,\Psi_0}{6\,M^2}$$
(4.35)

we have for the electric and magnetic field in tetrad components with respect to untilded coordinates (recalling that the quantities we previously calculated are the tilde ones),

$$\mathcal{E}_{\hat\varphi} = 0$$
$$\mathcal{E}_{\hat r} = F_{\hat t \hat r} = -\tilde A_{,r}$$
$$= -Q\,\tilde A_r^{(1)}\,[1 + C_A\,Q^2]$$
$$\mathcal{E}_{\hat\theta} = F_{\hat t \hat\theta} = \frac{\sqrt\delta\,\tilde A_{,\mu}}{r}$$
$$= \frac{\sqrt\delta\,\tilde A_\mu^{(1)}}{r}\,[1 + C_A Q^2]$$
(4.36)

and

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$$\mathcal{B}_{\hat{\varphi}} = 0$$

$$\mathcal{B}_{\hat{r}} = F_{\hat{\varphi}\hat{\theta}} = \frac{\sqrt{\delta}\, B_{,\mu}}{r^2 \sin\theta}$$

$$\approx -\frac{\sqrt{\delta}\Delta}{\tilde{\Psi}} \frac{e^{a\tilde{\phi}}\, Q \tilde{B}_r^{(1)}}{r^2 \sin\theta} [1 + C_B\, Q^2]$$

$$\mathcal{B}_{\hat{\theta}} = -F_{\hat{\varphi}\hat{r}} = \frac{B_{,r}}{r \sin\theta}$$

$$\approx \frac{\delta}{\tilde{\Psi}} \frac{e^{a\tilde{\phi}}\, \tilde{B}_\mu^{(1)}}{r \sin\theta} [1 + C_B\, Q^2] . \tag{4.37}$$

In the limit $r \to \infty$ the fields can be approximated as

$$\mathcal{E}_{\hat{r}} \approx \frac{Q_{phys}}{r^2}$$

$$\mathcal{E}_{\hat{\theta}} \approx -\frac{2\,\alpha^2}{r^4}\, Q_{phys}\, \sin\theta\, \cos\theta$$

$$\mathcal{B}_{\hat{r}} \approx \frac{2\,\alpha\, Q_{phys}}{r^3}\, \cos\theta \left[1 - \frac{a^2\, Q^2}{6\, M^2}\right]$$

$$\mathcal{B}_{\hat{\theta}} \approx \frac{\alpha\, Q_{phys}}{r^3}\, \sin\theta \left[1 - \frac{a^2\, Q^2}{6\, M^2}\right] . \tag{4.38}$$

where

$$Q_{phys} \equiv Q \left[1 + \frac{a^2\, Q^2}{3\, M^2}\right] , \tag{4.39}$$

and the hatted variables are the usual flat spacetime spherical coordinates [16]. If we square the latter relation, we find that to order Q^3, $Q_{phys}^2 = Q^2$.

To order Q^2 the dilaton field solution agrees with the following expression

$$\tilde{\phi} = -\frac{1}{a}\, \ln b^{2\gamma} \tag{4.40}$$

where

$$\gamma = \frac{2\, a^2}{1 + a^2} , \tag{4.41}$$

and

$$b^2 = 1 + \frac{Q^2\, (1 + a^2)\, M\, r}{2\, M^2\, \rho_0^2} . \tag{4.42}$$

When $a^2 = 3$ this form reproduces the Kaluza–Klein solution [10,11].

Also, our perturbative result for $\tilde{\Psi}$ and ρ agrees with the following expressions (up to order Q^2)

$$\tilde{\Psi} = -\frac{\Delta - \alpha^2\,\delta}{\rho^2}\ ,\tag{4.43}$$

and

$$\rho^2 = \rho_0^2\,e^{-a\,\tilde{\phi}/3}\ .\tag{4.44}$$

Similarly, the following expression

$$\tilde{\omega} = -\alpha\,\delta\,[1 + \tilde{\Psi}^{-1}]\ ,\tag{4.45}$$

reproduces our perturbative results to order Q^2.

Let us notice further that the following expressions for the potentials $\tilde{A}$ and $\tilde{B}$ are again in agreement with our perturbative calculation to order Q^2

$$\tilde{A} = Q_{phys}\,\frac{r}{\rho^2}\,e^{a\,\tilde{\phi}/3}\ ;\quad \tilde{B} = -Q_{phys}\,\frac{\alpha\,\mu}{\rho^2}\,e^{-a\,\tilde{\phi}/3}\left(1 - \frac{a^2\,Q^2}{6\,M^2}\right)\ .\tag{4.46}$$

The covariant and contravariant metric tensor elements to the same order in Q are (we have reinstated the tilde)

$$g_{tt} = \tilde{\Psi}$$
$$\approx -\left[1 - \frac{2M}{r}\left(1 + \frac{Q^2\,\lambda_0}{2M^2}\right)\right]$$
$$g^{tt} = \frac{1}{\tilde{\Psi}} - \frac{\tilde{\omega}^2\,\tilde{\Psi}}{\Delta\delta}$$
$$\approx -\left[1 + \frac{2M}{r}\left(1 + \frac{Q^2\,\lambda_0}{2M^2}\right)\right]$$
$$g_{\varphi\varphi} = \tilde{\Psi}\,\tilde{\omega}^2 - \frac{\Delta\delta}{\tilde{\Psi}}$$
$$\approx r^2\,\delta$$
$$g^{\varphi\varphi} = -\frac{\tilde{\Psi}}{\Delta\,\delta}$$
$$\approx \frac{1}{r^2\delta}$$
$$g_{t\varphi} = \tilde{\Psi}\,\tilde{\omega}$$

$$\approx \frac{-2\alpha\,\delta\,M}{r}\left(1+\frac{Q^2\,\lambda_0}{2\,M^2}\right)$$

$$g^{t\varphi} = \frac{1}{\Delta\,\delta}\,\tilde{\Psi}\,\tilde{\omega}$$

$$\approx \frac{-2\alpha\,M}{r^3}\left(1+\frac{Q^2\,\lambda_0}{2\,M^2}\right)$$

$$g_{rr} = \frac{\rho^2}{\Delta}$$

$$\approx 1+\frac{2M}{r}\left(1+Q^2\,\frac{\lambda_0}{2\,M^2}\right)$$

$$g^{rr} = \frac{\Delta}{\rho^2}$$

$$\approx 1-\frac{2\,M}{r}\left(1+\frac{Q^2\lambda_0}{2\,M^2}\right)$$

$$g_{\theta\theta} = \rho^2$$

$$\approx r^2$$

$$g^{\theta\theta} = \rho^{-2}$$

$$\approx r^{-2}\,, \tag{4.47}$$

where $\approx$ denotes again the $r \to \infty$ approximation.

V. CORRECTIONS TO THE GYROMAGNETIC RATIO

The gyromagnetic ratio for the Kerr-Newman black hole is known to be $g = 2$ [15]. The dilaton component of the gravitational field shifts g away from 2. The shift has been calculated for the case of small black hole angular momentum [10]. Using the expressions for the $g_{t\varphi}$ component of the tensor and the $\mathcal{B}_\theta$ component of the magnetic field, we can calculate the gyromagnetic ratio for arbitrary angular momentum. The magnetic moment measured by a distant observer is

$$\mu_{phys} = g\,\frac{Q_{phys}\,J_{phys}}{2\,M_{phys}}\,, \tag{5.1}$$

where Q_{phys} is given in Eq.(4.39) and μ_{phys}, J_{phys} and M_{phys} are obtained from the asymptotic expressions for $\mathcal{B}_\theta$, $g_{t\varphi}$ and g_{tt} respectively

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$$\mu_{phys} = \alpha \, Q \left[1 + \frac{a^2 \, Q^2}{6 \, M^2} \right]$$

$$J_{phys} = \alpha \, M \left[1 + \frac{a^2 \, Q^2}{6 \, M^2} \right]$$

$$M_{phys} = M \left[1 + \frac{a^2 \, Q^2}{6 \, M^2} \right] \tag{5.2}$$

and, as we stated previously, all corrections are proportional to the $(Q/M)^2$ ratio. The gyromagnetic ratio obtained from these expressions is

$$g = 2 \left[1 - \frac{a^2 \, Q_{phys}^2}{6 \, M_{phys}^2} \right] \, . \tag{5.3}$$

This agrees with the expression obtained in Ref. [10] when their expression is expanded to order Q^2. Our expression differs from theirs in that the ansatz they chose for the metric assumes there is no dilatonic correction to the physical mass and the charge. Our solutions contain no assumptions, other than requiring that our solutions give the Kerr–Newman solutions in the $a = 0$ limit, and predict that both the charge and the mass of the black hole are modified by the presence of the dilaton.

VI. DISCUSSION

The solutions obtained here for the fields should be very useful in determining the effects of a scalar gravitational field on physically measurable quantities. In our derivation we have made no assumptions about the form of the metric tensor elements other than the small charge-to-mass ratio of the black hole. In particular our solutions are valid for black holes with arbitrary angular momentum and dilatonic charge a, whereas previous solutions have required that the black holes have small angular momentum or specific values of a. These are important distinctions, because a black hole can in principle have appreciable angular momentum. Having a as an arbitrary parameter is, of course, a desirable feature of any measurable quantity obtained for this geometry.

Our solutions provide a framework for the calculation of physically measurable quantities, such as radiation intensities for radiation emitted from the region near a black hole horizon.

Since the wave equation and Maxwell's equations are separable in the Kerr–Newman geometry [16], it may be possible to obtain analytic expressions for the wave functions, at least in some linear approximation of the gravitational field. We are presently carrying out these calculations. The goal of these calculations is to provide measurable predictions of the effect of the scalar component of gravity.

Having explicit expressions for the metric tensor elements will allow us to carry out an analysis of the statistical mechanics of a gas of Kerr–Newman dilatonic black holes as we have done previously for the Schwarzschild, Reissner-Nordstrøm and dilaton black holes [1]. In particular it will be interesting to see if a gas of such black holes obeys the bootstrap relation.

Finally, we note here that our solutions possess a mirror symmetry. This symmetry has been studied extensively within the context of string theory, where it has been conjectured to be a property of the Calabi-Yau space associated with a (2,2) conformal field theory. It may then be an interesting problem to generalize the present work to the case where both a dilaton and an axion are present.

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